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PROPAGATION OF SINGULARITIES, HAMILTON-JACOBI EQUATIONS AND NUMERICAL APPLICATIONS

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ABSTRACT. We consider applications of Hamilton-Jacobi equations for which the initial data is only assumed to be in L^∞. Such problems arise for example when one attempts to describe several characteristic singularities of the compressible Euler equations such as contact and acoustic surfaces, propagating from the same discontinuous initial front. These surfaces represent the level sets of solutions to a Hamilton-Jacobi equation which belongs to a special class. For such Hamilton-Jacobi equations we prove the existence and regularity of solutions for any positive time and convergence to initial data along rays of geometrical optics at any point where the gradient of the initial data exists. Finally, we present numerical algorithms for efficiently capturing singular fronts with complicated topologies such as corners and cusps. The approach of using Hamilton-Jacobi equations for capturing fronts has been used in [14] for fronts propagating with curvature-dependent speed.

1. INTRODUCTION

In this paper we consider a class of Hamilton-Jacobi equations

$$u_t + H(x, t, u, Du) = 0, \quad Du = (u_{x_1}, u_{x_2}, \dots, u_{x_n}) \tag{1}$$

with L^∞ initial data and their applications to the propagation of singularities for nonlinear hyperbolic equations such as the compressible Euler equations. For smooth initial data (C^1 or better), the geometrical optics construction yields a classical solution of (1) which is valid for a short time [2]. In the early 1980's, Crandall and Lions and Crandall, Evans and Lions developed the theory for weak solutions which are only uniformly continuous [3, 4]. They introduced the class of viscosity solutions and proved the existence and uniqueness of global weak solutions in that class. In earlier work, S. N. Kruzhkov considered the special case where the hamiltonian $H(x, t, u, p)$ was a convex function of p and proved the existence and uniqueness of weak solutions to the Dirichlet problem in the class of functions that satisfy $D^2u/Dl^2 \geq -c$, for some c, in a weak sense for any direction l [8]. This condition is analogous to the one-sided Lipschitz condition for convex conservation laws [16].

We are motivated by a way of describing the propagation of characteristic singularities in solutions of the compressible Euler equations; this is the subject

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of our next section. The problem leads naturally to the consideration of a class of Hamilton-Jacobi equations with discontinuous initial data; more generally, we consider L^∞ initial data. The class of hamiltonians under consideration satisfy the following condition which is crucial in proving a priori estimates for existence: There exists $d > 0$ so that for some positive constants C, R:

$$H_u(x, t, u, p) \geq C|p|^{2d}, \qquad |p| \geq R, \ |u| \leq M, \tag{2}$$

where the constant M should be the L^∞ norm of the initial data.

We remark that the hamiltonian need not be convex, in p, and in our application it is, in fact, not convex. Hence the estimates in Kruzhkov [8] do not apply.

Using this condition we will be able to obtain an estimate on $|Du^\varepsilon(x, t)|$, $t > 0$, depending only on t and not on derivatives of the initial data or ε. Here u^ε represents the solution to a viscous regularization of equation (1). This estimate combined with the standard L^∞ estimate on u and an additional argument that gives regularity in t is enough to guarantee that a subsequence converges to a viscosity solution of (1) for $t > 0$. Clearly, we cannot expect this estimate to hold uniformly as t tends to zero, so one needs to discuss the way in which the solution attains its initial values u_0. We will show that at any point x_0 where $Du_0(x_0)$ exists, $u(x(t), t) \to u_0(x)$ as $t \to 0$, where $x(t)$ is the curve obtained from the geometrical optics construction such that $x(0) = x_0$.

Questions regarding uniqueness of solutions (with L^∞ initial data) remain open.

2. Propagation of singularities in the compressible Euler equations

We consider the initial value problem for the compressible Euler equations where the initial data is piecewise smooth and discontinuous along a smooth compact surface S.

The compressible Euler equations are a hyperbolic system of five conservation laws:

$$\begin{gathered} \rho_t + \operatorname{div}(\rho q) = 0, \\ (\rho q)_t + \operatorname{div}(\rho q q^T) + D_x p = 0, \\ E_t + \operatorname{div}(q(E + p)) = 0, \end{gathered}$$

where $\rho =$ density, $q =$ the velocity in 3D, $E =$ total energy, $p = p(\rho, q, E)$ $=$ pressure. The equations express the conservation of mass, momentum and energy in fluid flow.

One expects the solution for small positive time to be piecewise smooth, and depending on the initial data, its singularities to consist of one contact discontinuity surface, or slip surface, and other surfaces that could be either shocks, rarefactions or just acoustic surfaces. The justification for this is provided by looking at the following much simpler problem (see [7]): Suppose we introduce spatial coordinates which are normal and tangential to S, locally near S, and regard the new partial differential equations in the new coordinates. If we ignore the tangential derivatives, we obtain a one-space dimensional system of conservation laws:

$$U_t + F(U, N(y))_x = 0$$

where $U = (\rho, \rho q, E)$, F = known flux vector, N = normal vector to S, x, y = normal, tangential coordinates. Consider the Riemann problem (with y as a parameter) for this system with initial states given by the restriction of the initial data on S from one side to the other. If the jump at S is small enough this Riemann problem can be solved (see [10]) and the solution is seen to consist of shocks, rarefactions and contact singularities. The singularities correspond to the eigenvalues of the jacobian matrix F_U. For the Euler equations F_U has three distinct real eigenvalues: $q \cdot N - c$, $q \cdot N$, $q \cdot N + c$, where $c = \sqrt{p_\rho}$ is the speed of sound. The contact discontinuity corresponds to the middle eigenvalue and it propagates with the eigenvalue speed i.e. it is characteristic. Shocks and rarefactions correspond to the other eigenvalues. Rarefactions consist of fans of rays emanating from the initial discontinuity, each ray propagating with eigenvalue speed. Shocks are discontinuities which are not characteristic but close to being characteristic when the shocks are weak.

The work of A. Majda [11, 12] gives the existence of a smooth multidimensional shock front for a short time, that is, assuming the Riemann problem consists of only one shock everywhere along S, then there exists a weak solution to the multidimensional problem consisting of only one shock surface. The existence of a smooth rarefaction was proved by Alhinac [1]. More generally, in [7] it was shown that assuming piecewise analytic initial data, for a short time there exists a piecewise analytic solution with the same general wave pattern (shocks, rarefactions and contacts) as obtained by solving the one-dimensional Riemann problem along S. The solution may contain acoustic waves where the solution is continuous but its derivatives in general are discontinuous and the surfaces are characteristic.

In this discussion we are interested in the case when the singularities propagating from the initial discontinuous front S consist only of characteristic surfaces such as rarefactions, contacts or acoustic surfaces. For simplicity we consider the case in which there is a contact surface and two acoustic surfaces—one ahead and the other behind the contact. Physically, this is the situation where the contact surface dominates the flow and the shocks are very weak and well approximated by acoustic waves. If the velocity of the flow is given by $q(x, t)$, and the speed of sound by $c(x, t)$ we have the following equations describing the evolution of the three surfaces separately:

$$X_t^0(a, t) = q(X^0(a, t), t), \tag{3}$$
$$X_t^{\pm 1}(a, t) = q(X^{\pm 1}(a, t), t) \pm c(X^{\pm 1}(a, t), t)N(X^{\pm 1}(a, t))$$

where X^0 describes the contact, $X^{\pm 1}$ the acoustic surfaces, $X^0(a, 0) = X^{\pm 1}(a, 0)$ is a parameterization of the initial front S, and N is the normal vector to the surface. The velocity q jumps across the surface $X^0(a, t)$ but since the normal component of q is continuous for a contact one can define $q(X^0(a, t), t)$ to be the restriction of q on X^0 from either side of the surface. In order to be able to define the normal $N(X)$, the surface S should be at least C^1 regular.

It is well known that any one surface given above is the characteristic surface for an eiconal equation [2]. This is a Hamilton-Jacobi equation which does not satisfy (2). In fact, in this case H_u is always 0. The initial data must be continuous in this case in order to get the existence of a weak solution. From a numerical point of view it has been pointed out that there are great advantages to capturing these surfaces by very simple and efficient finite difference methods for the eiconal equation [14, 5]. With this approach one can handle nonsmooth surfaces, for example surfaces with corners and cusps for which the above Lagrangian formulation fails.

In this section our goal is to derive a single Hamilton-Jacobi equation whose characteristic surfaces include the three surfaces emerging from S where they intersect at time 0. This will be an equation that satisfies (2). In this case it is very natural to consider initial data which is discontinuous because if each surface represents a different level set of the solution u, then u must be discontinuous where the surfaces intersect. Let us consider the family of surfaces $X^s(a, t)$, parametrized by $s \in [-1, 1]$ which satisfy

$$X_t^s(a, t) = q(X^s(a, t), t) + sc(X^s(a, t), t)N(X^s(a, t)).$$

It is easy to show that for small enough $t > 0$ the map that takes (a, s) into $X^s(a, t)$ defines two diffeomorphisms; one that takes $S \times [-1, 0]$ onto Ω_t^- which is the closed region in R^3 bounded by one acoustic surface and the contact surface at time t and the other that takes $S \times [0, +1]$ onto Ω_t^+ which is the region bounded by the contact and the other acoustic surface. The map $X^s(a, t)$ is in general discontinuous at the slip surface $s = 0$. The fact that these maps are diffeomorphisms for small enough t is verified easily by checking that the jacobian derivative $D_{s,a}X$ is a nonsingular for all small enough positive t. In addition one can easily show that if for each s, $X_t^s(a, t)$ are smooth functions of a for $t \in [0, T)$ then they define a diffeomorphism for any t in $(0, T)$. The argument here is that if the surfaces intersect at some time $t > 0$ for two different values of s, say $s_1 < s_2$ then, since this cannot happen arbitrarily close to $t = 0$ there must be a smallest strictly positive time t_0 when it happens. But then, at this first time, the surfaces must intersect tangentially since the set of t where the surfaces intersect transversally is open. Suppose X_0 with coordinate a_0 is one such intersection point. Then

$$(X_t^{s_1}(a_0, t_0) - X_t^{s_2}(a_0, t_0)) \cdot N(X_0) = (s_1 - s_2)c(X_0, t_0) < 0$$

and this implies that the surfaces must intersect at some time $t < t_0$ which is a contradiction.

We can now define the function $u(x, t)$ implicitly by

$$u(X^s(a, t), t) = s.$$

It follows easily now that u is a smooth function in $\Omega_t^- \cup \Omega_t^+$ of x, t and by differentiating in t we find that it satisfies the Hamilton-Jacobi equation

$$u_t + q(x, t) \cdot Du + c(x, t)u|Du| = 0. \tag{4}$$

At time $t = 0$, if we let u be discontinuous across S and defined to be -1 on one side and $+1$ on the other then, at any later time, the contact is given by $u = 0$ and the acoustic waves are given by $u = \pm 1$ in Ω_t. The condition in (2) is satisfied with $d = \frac{1}{2}$.

In the next section we will show that for Lipschitz continuous functions q and c one can prove the existence of viscosity solutions of (4) with L^∞ initial data. The existence does not depend on how large $|D_x q|$, $|D_x c|$ are so one can apply this to arbitrarily thin shear layers where the tangential velocity is continuous but changes very rapidly within a very thin layer.

In addition to having numerical applications, the equation (4) offers a very nice change of coordinates for the discontinuous initial value problem for the compressible Euler equations so that the geometry becomes simple and free boundaries are eliminated. In fact this was our motivation at the start.

Finally, it helps to consider the model equation obtained by considering (4) in one space dimensions with $q \equiv 0$, and $c \equiv 1$:

$$u_t + u|u_x| = 0.$$

If we consider the initial data given by the function which is constant $+1$ or -1 in $x < 0$ and $x > 0$ but jumps at 0 then one easily verifies that a viscosity solution is given by a rarefaction wave no matter which way the initial data jumps. One could have anticipated this since the equation is invariant under reflections.

The same model equation but with time going backwards (which is equivalent to changing the $+$ into a $-$ in the equation) was considered by L. Rudin [15] and Osher and Rudin [13] in connection with the question of how to "sharpen pictures" for image processing.

3. Existence of solutions and convergence to the initial data

In this section we obtain estimates for the solutions to viscous approximations of (1) and as a result show that there is a subsequence converging to a viscosity solution of (1). We also consider the geometrical optics construction as a tool to show that the initial data is attained at any point where it is differentiable.

We consider hamiltonians $H(x, t, u, p)$ which are continuous, Lipschitz in x for $|p| \geq R$ and in addition to (2) they satisfy

$$\begin{gathered} \operatorname*{ess\,sup}_x |D_x H| \leq \frac{C}{2}|p|^{1+2d}, \qquad |p| \geq R,\ |u| \leq M, \\ |H| \leq C_0(1+|p|^2), \quad H(x, t, u, 0) = 0, \end{gathered} \tag{5}$$

where the constants C, d, R, M are the same as in (2).

Theorem 1. *Let u satisfy*

$$u_t + H(x, t, u, Du) = \varepsilon \Delta u \tag{6}$$

with initial data $u(x, 0) = u_0(x)$ which is twice continuous differentiable, constant outside some compact set and which is bounded by M. Then there exists a solution $u(x, t)$ such that for $t > 0$

$$|u|_\infty \leq M, \quad |Du|_\infty(t) \leq \max\left(\sqrt{R}, \left(\frac{1}{C\,dt + \frac{1}{|Du_0|_\infty^{2d}}}\right)^{1/2d}\right). \tag{7}$$

Remark. The hamiltonian in (4) is $H = q \cdot p + uc|p|$ so (2) holds with $C = \min_{x,t} c(x, t) > 0$, $d = \frac{1}{2}$ and R arbitrary. Condition (5) is fulfilled if we choose $R = 2(|D_x q|_\infty + M|c_x|_\infty)/C$, where $|D_x q|^2 = \sum_i |q_{x_i}|^2$.

Proof of Theorem 1. The conditions imposed on the hamiltonian H ensure the existence of a C^2 solution and the L^∞ bound. This follows from standard results on parabolic equations [9]. Therefore, the only thing to prove is the estimate (7).

We assume without loss of generality that H is continuously differentiable everywhere since otherwise we can approximate it by a smooth H and use the fact that the estimates do not depend on this approximation. Let $u_i = D_{x_i}u$. We differentiate (6) with respect to x_i,

$$(u_i)_t + H_{x_i} + H_u u_i + H_p \cdot Du_i = \varepsilon \Delta u_i ,$$

multiply by $2u_i$,

$$(u_i^2)_t + 2u_i H_{x_i} + 2H_u u_i^2 + H_p \cdot Du_i^2 = \varepsilon(\Delta(u_i^2) - 2|Du_i|^2),$$

and sum from $i = 1$ to n,

$$(|Du|^2)_t + 2Du \cdot D_x H + 2H_u|Du|^2 + H_p \cdot D|Du|^2 = \varepsilon \left(\Delta|Du|^2 - 2\sum_i |Du_i|^2 \right).$$

Let $K(t)$ be the set of points in R^n where $|Du|(\cdot, t)$ attains its maximum. For each t, $K(t)$ is closed and since $|Du|$ tends to zero at infinity uniformly the union of $K(t)$, $0 \le t \le T_0$ is bounded. At any point of $K(t)$, $D|Du|^2 = 0$, and $\Delta|Du|^2 \le 0$ and therefore the following inequality holds:

$$(|Du|^2)_t(x, t) \le -2H_u|Du|^2(x, t) + 2|D_x H|\,|Du|(x, t). \tag{8}$$

Let $z(t) = \sup_x |Du|^2(x, t)$. One has the following lemma which will be proved later:

Lemma 1. *The function $z(t)$ is differentiable and for some $x \in K(t)$*

$$z'(t) = \frac{d|Du|^2}{dt}(x, t).$$

Assuming the lemma, it follows from (2), (5) and (8) that if $z \ge R$ on some interval $[t_0, t]$ then everywhere in that interval $z' \le -Cz^{1+d}$ or $(z^{-d})' \ge Cd$. Hence, by integrating from t_0 to t we get

$$z(t) \le \left(\frac{1}{Cd(t - t_0) + z^{-d}(t_0)} \right)^{1/d}. \tag{9}$$

We claim that if $z(t_0) \le R$ for some $t_0 \ge 0$ then $z(t) \le R$ for all $t \ge t_0$. Otherwise, since z is continuous there must be an interval $[T, T+\delta]$ so that $z(T) = R$ and $z > R$ in $(T, T+\delta]$. But using (9) we get $z(T+\delta) < R$ which is a contradiction.

Consider now the case when $z(0) > R$ and let $[0, t_1)$ be the largest interval on which $z(t) \ge R$. On this interval the estimate (9) is valid. If t_1 is infinity then the estimate (9) holds with $t_0 = 0$ and any t so (7) follows. If t_1 is finite then $z(t_1) = R$ and since in this case we showed that $z(t) \le R$ for $t \ge t_1$, proof of (7) is complete.

Remark. If $R = 0$ the theorem gives a rate of decay estimate for $|Du|_\infty(t)$.

Proof of Lemma 1. We fix $t > 0$ and let $x_0 \in K(t)$, $x_1 \in K(t+\delta)$ where δ is small enough so that $t + \delta < 0$. Then

$$\begin{aligned} z(t+\delta) - z(t) &= u(x_1, t+\delta) - u(x_0, t) \\ &= u(x_1, t+\delta) - u(x_1, t) + u(x_1, t) - u(x_0, t) \\ &= \frac{du}{dt}(x_1, t)\delta + u(x_1, t) - u(x_0, t) + O(\delta^2) \end{aligned} \tag{10}$$

since u is C^2. In addition,

$$0 \le u(x_0, t) - u(x_1, t) \le u(x_0, t) - u(x_0, t+\delta) + u(x_1, t+\delta) - u(x_1, t)$$
$$= -\frac{du}{dt}(x_0, t)\delta + \frac{du}{dt}(x_1, t)\delta + O(\delta^2).$$

Hence, from (10)

$$z(t+\delta) - z(t) \le \frac{du}{dt}(x_1, t)\delta + O(\delta^2), \tag{11}$$

$$z(t+\delta) - z(t) \ge \left(2\frac{du}{dt}(x_1, t) - \frac{du}{dt}(x_0, t)\right)\delta + O(\delta^2). \tag{12}$$

This shows that z is continuous. We now claim that for a fixed t there exists $x_0 \in K(t)$ so that $\operatorname{dist}(x_0, K(t+\delta) \to 0$ as $\delta \to 0$. If this were not true then we could find $\delta_n \to 0$ and $x_n \in K(t+\delta_n)$ so that $\operatorname{dist}(x_n, K(t)) \ge \varepsilon > 0$ for all n. Since x_n are bounded, by choosing a subsequence if necessary we can assume that $x_n \to x_*$. So now $z(t+\delta_n) = u(x_n, t+\delta_n) \to u(x_*, t) < z(t)$ since x_* is not in $K(t)$. Since z is continuous, this is a contradiction.

Hence we can choose $x_1(\delta) \in K(t+\delta)$ so that $x_1(\delta) \to x_0$ as $\delta \to 0$. It now follows from (11) and (12) that

$$\lim_{\delta\to 0} \frac{z(t+\delta) - z(t)}{\delta} = \frac{du}{dt}(t, x_0)$$

and the lemma follows.

Remark. During the writing of this paper we discovered that getting the estimate (7) from inequalities such as (8) is known in the literature as Bernstein's method [6].

We now consider the initial value problem for the Hamilton-Jacobi equations (1) where $u(x, 0) = u_0(x) \in L^\infty$ and it is equal to a constant outside some compact set. Let $u_{0_\delta} = u_0 * j_\delta$ where j is a smooth mollifier so that $u_{0_\delta} \to u_0$ almost everywhere. Let us denote by $u_\delta^\varepsilon(x, t)$ the solution of (6) with initial data given by u_{0_δ}. By Theorem 1 and the Arzela-Ascoli Theorem, for any sequence of ε's there exist a subsequence that converges uniformly on compact subsets of R^n to u_δ for all rational $t > 0$. Furthermore the estimate in Theorem 1 holds for u_δ and it is uniform in δ so for any sequence of δ's one can find a subsequence which converges for all rational $t > 0$ to a function u.

We will show that we can extend u to be defined for all t and that it is Lipschitz continuous in $R^n \times (0, \infty)$. Moreover, if we redefine u_δ^ε to denote the subsequence which converges to u for all rational t, then $u_\delta^\varepsilon \to u$ uniformly everywhere on compact subsets of $R^n \times (0, \infty)$. In order to prove this let $0 < t_0 < t_1$ be two rational points. If we convolve equation (6) with j_η and integrate from T_0 to t_1 we get

$$|u_\delta^\varepsilon * j_\eta(x, t_1) - u_\delta^\varepsilon * j_\eta(x, t_0)|$$
$$\le \int_{t_0}^{t_1} |H(x, t, u_\delta^\varepsilon, Du_\delta^\varepsilon) * j_\eta(x, t)|\,dt + \varepsilon \int_{t_0}^{t_1} |u_\delta^\varepsilon * \Delta j_\eta(x, t)|\,dt.$$

If we let ε, δ tend to 0 and then η tend to zero, in view of Theorem 1 we get

$$|u(x, t_1) - u(x, t_0)| \le \text{const}(t_1 - t_0)$$

so we can extend u to a Lipschitz function for all $t > 0$.

Now consider any positive irrational t and let t_* be an arbitrarily close rational. Then

$$|u_\delta^\varepsilon(x, t) - u(x, t)| \le |u_\delta^\varepsilon(x, t) - u_\delta^\varepsilon * j_\eta(x, t)| + |u_\delta^\varepsilon * j_\eta(x, t) - u_\delta^\varepsilon * j_\eta(x, t_*)| + |u_\delta^\varepsilon * j_\eta(x, t_*) - u(x, t)|.$$

The first term on the right side of the inequality is bounded by $\eta|Du_\delta^\varepsilon|_\infty$. For a fixed η, as ε, δ tend to 0, following an argument similar to the one above we find that the second term is bounded by $\text{const}\,|t-t_*|+o(1)$. Similarly the third term is bounded by $|u * j_\eta(x, t_*) - u(x, t_*)| + \text{const}\,|t - t_*| + o(1)$. Therefore

$$\limsup_{\varepsilon, \delta\to 0} |u_\delta^\varepsilon(x, t) - u(x, t)| \le \text{const}(\eta + |t - t_*) + |u * j_\eta(x, t_*) - u(x, t_*)|.$$

Since η, $|t - t_*|$ are arbitrarily small and $u(x, t_*)$ is continuous in x,

$$\lim_{\varepsilon, \delta\to 0} |u_\delta^\varepsilon(x, t) - u(x, t)| = 0.$$

Moreover, the convergence is uniform on compact subsets of $R^n \times (0, \infty)$. Also, it is evident from the proof that the convergence does not depend on which order we take the limit $\varepsilon, \delta \to 0$.

It now follows from results in [3] that u is a viscosity solution of (1) and in particular u is a classical solution in any region where it has a bounded derivative. We refer the reader to [3] for the definition of a viscosity solution.

It remains to discuss in what sense does u attain its initial values u_0 since the estimate (7) is not useful when $|Du_0|_\infty = \infty$ and $t \to 0$. For this we will make use of the geometrical optics construction used in the classical theory of Hamilton-Jacobi equations.

If the initial data u_0 for (1) is continuously differentiable then one can obtain the solution locally by solving the following ODE's from the classical geometrical optics construction [2]:

$$u_s = H_p \cdot p + p_0, \quad p_s = -(H_u p + H_x), \quad p_{0_s} = -(H_u p_0 + H_t), \quad x_s = H_p,$$
$$u(0) = u_0(x_0), \quad p_0(0) = u_t(0, x_0), \quad p(0) = Du_0(x_0), \quad x(0) = x_0.$$

Suppose now that u_0 is only in L^∞ and without loss of generality let $x_0 = 0$ be the point where it is differentiable and that $u_0(0) = 0$, $Du_0(0) = 0$. We want to show that $u(x(s), s) \to 0$ as $s \to 0$ where u is the solution of (1) with initial data u_0.

Let us consider the approximations u_δ which are solutions to equation (1) with $u_\delta(x, 0) = u_0 * j_\delta$ and such that $u_\delta \to u$ for some sequence of $\delta \to 0$. By hypothesis, there exists an r such that $|u_0(x)| \le c_0|x|^2$, for $|x| < r$. One easily obtains that for some δ_0 and any δ from our sequence which is $\le \delta_0$

$$|u_\delta(x, 0)| \le c_1(|x|^2 + \delta),$$

for $|x| < r/2$. We can define a continuously differentiable function $L_\delta(x, 0) \le u_\delta(x, 0)$ which is equal to $-c_1(|x|^2 + \delta)$ for $|x| < r/4$, and it converges uniformly as $\delta \to 0$ to some function which we denote by $L(x, 0)$. Similarly, we can define $U_\delta(x, 0) \ge u_\delta(x, 0)$, $U_\delta(x, 0) = c_1(|x|^2 + \delta)$ for $|x| < r/4$, which is continuously differentiable and converges uniformly to $U(x, 0)$. Note that $L(0, 0) = U(0, 0) = 0$, and $D_x L(0, 0) = D_x U(0, 0) = 0$. Since $L(x, 0)$,

$U(x, 0)$ are continuously differentiable near 0, from this initial data one can construct the solutions locally using geometrical optics. Since the solution $L(x, t)$ is determined along the geometrical optics curve starting from 0 solely by $L(0, 0)$ and $D_x L(0, 0)$, and similarly for $U(x, t)$, for small enough s, $L(x(s), s) = U(x(s), s)$. Since $L_\delta(x, 0) \le u_\delta(x, 0) \le U_\delta(x, 0)$ by the monotonicity of the solution operator [3] it follows that

$$L_\delta(x, t) \le u_\delta(x, t) \le U_\delta(x, t).$$

Letting $\delta \to 0$ for any x and $t \ge 0$ we obtain

$$L(x, t) \le u(x, t) \le U(x, t)$$

and hence

$$u(x(s), s) = L(x(s), s) = U(x(s), s),$$

for small enough s. Therefore $u(x(s), s) = L(x(s), s) \to 0$ as $s \to 0$.

4. Numerical algorithms and experiments

In this section we present a numerical algorithm for solving the model equation

$$u_t + u|Du| = 0 \tag{13}$$

which is obtained from (4) by setting $q = 0$ and $c = 1$. The level surfaces of (13) move in the direction normal to themselves with speed equal to u. For simplicity we restrict our attention to two-space dimensions. The purpose of this section is to show that a very simple finite difference algorithm is capable of capturing topologically complicated level curves of u. Other similar schemes have been used by Osher and Sethian in [14]. A more systematic study of numerical algorithms approximating (4) including stability estimates and convergence will be included in another article.

We approximate the solution $u(t, x, y)$ on a grid by $u(t_n, x_j, y_k) = u^n_{jk}$ and use the following notation for forward and backward difference operators:

$$\Delta^x_+ u_{jk} = u_{j+1,k} - u_{jk}, \quad \Delta^x_- u_{jk} = u_{jk} - u_{j-1,k},$$
$$\Delta^y_+ u_{jk} = u_{j,k+1} - u_{jk}, \quad \Delta^u_- u_{jk} = u_{jk} - u_{j,k-1}, \quad \Delta^t u^n = u^{n+1} - u^n.$$

We would like to design an upwind scheme using a procedure similar to the one used for conservation laws (see [14] for references). Let us consider the 1d equation first

$$u_t + u|u_x| = 0$$

or written differently

$$u_t + \operatorname{sign}(u_x) u u_x = 0.$$

The upwind direction is determined by the sign of uu_x and this determines whether we use a forward or a backward difference formula to approximate space derivatives. In fact one would like to find an effective "splitting" of the space derivatives into its "upwind" and "downwind" parts. One possibility which generalizes easily to multidimensions is given by the following scheme

$$\Delta^t u + \operatorname{sgn}(u)((u\Delta^x_- u)\operatorname{pos}(u\Delta^x_- u) + (u\Delta^x_+ u)\operatorname{neg}(u\Delta^x_+ u))^{1/2} = 0$$

where $u = u_j^n$ and pos, neg denote the positive and negative parts. The 2d scheme is given by

$$\Delta^t u + \operatorname{sgn}(u)((u\Delta_-^x u)\operatorname{pos}(u\Delta_-^x u) + (u\Delta_+^x u)\operatorname{neg}(u\Delta_+^x u) + (u\Delta_-^y u)\operatorname{pos}(u\Delta_-^y u) + (u\Delta_+^y u)\operatorname{neg}(u\Delta_+^y u))^{1/2} = 0$$

where in this case $u = u_{jk}^n$.

This scheme is very similar to a scheme introduced by Osher and Rudin [13] for an equation used for image enhancement and as remarked there, it would reduce to the Godunov scheme if u (in front of $|u_x|$) was replaced by a fixed constant.

This scheme is first order accurate but it can be easily extended to a second order accurate scheme in space by the following procedure: It is enough to consider the 1d case since the same procedure is applied in each coordinate separately. Define

$$s_- = \operatorname{minmod}(\Delta_-^x u_j - \Delta_-^x u_{j-1}, \Delta_+^x u_j - \Delta_-^x u_j),$$
$$s_+ = \operatorname{minmod}(\Delta_+^x u_j - \Delta_-^x u_j, \Delta_+^x u_{j+1} - \Delta_+^x u_j)$$

where $\operatorname{minmod}(a, b)$ is the number with the minimum modulus if the two numbers have the same sign and 0 otherwise. Now we replace $\Delta_-^x u$ and $\Delta_+^x u$ in our first order scheme by $\Delta_-^x u + s_-/2$ and $\Delta_+^x u - s_+/2$ respectively. This is the standard MUSCL approach due to Van-Leer developed originally for conservation laws. The resulting scheme is second order accurate away from extrema of u_x, u_y.

4.1 **Computational experiments.** For computational experiments we choose examples which contain two interesting kinds of singularities in curves: corners and cusps. These are examples for which methods based on discretizing the equations (3) run into major difficulties for obvious reasons (see [14]).

Our computations were performed using the second order scheme we described before.

We first computed solutions to (13) with initial data given by the function which is $+1$ in the interior of a square and -1 outside. The solution at some positive time and its level curves are shown in Figure 1. In the top picture the bottom part of the graph represents the level $u = -1$ and the top part $u = +1$. The level curve $u = 0$ is the square which is exactly in the middle (the fourth level curve from the outside or from the inside) and this is in fact the square where originally the initial data had the jump. The level curve $u = -1$ is the square with its corners rounded which is the outermost curve in the picture. The level curve $u = 1$ is the innermost square and is continuously shrinking. This is the correct viscosity solution.

Next we considered the initial data given by the function which is $+1$ in the interior of two tangent circles and -1 outside this region which is a figure 8 configuration (Figure 2). Figure 3 shows the solution and the level curves at some positive time. As before the level curve $u = 0$ is exactly in the middle (the fourth one from the outside or from the inside) and it is the original figure 8 where the initial data had a jump. The level curve $u = 1$ is given by the two circles in the middle which became separated after $t = 0$ and are continuously shrinking. The level curve $u = -1$ is the outermost curve consisting of parts of two big circles intersecting at the two sharp corners on the left and right. This again is the correct viscosity solution.

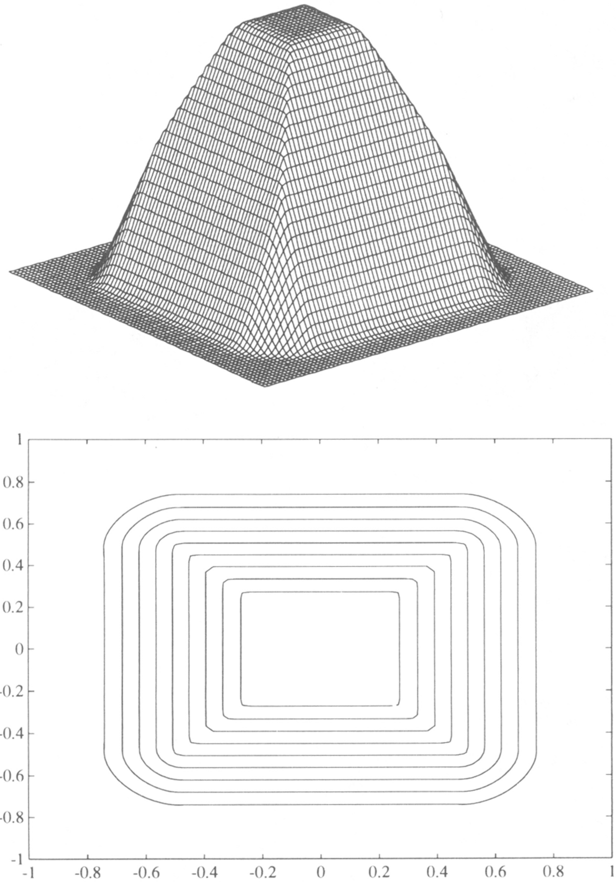

FIGURE 1

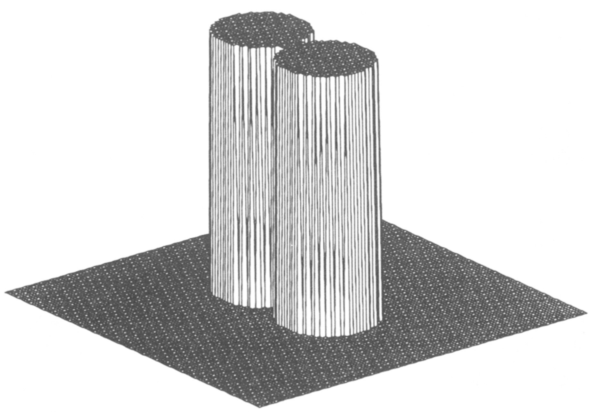

FIGURE 2

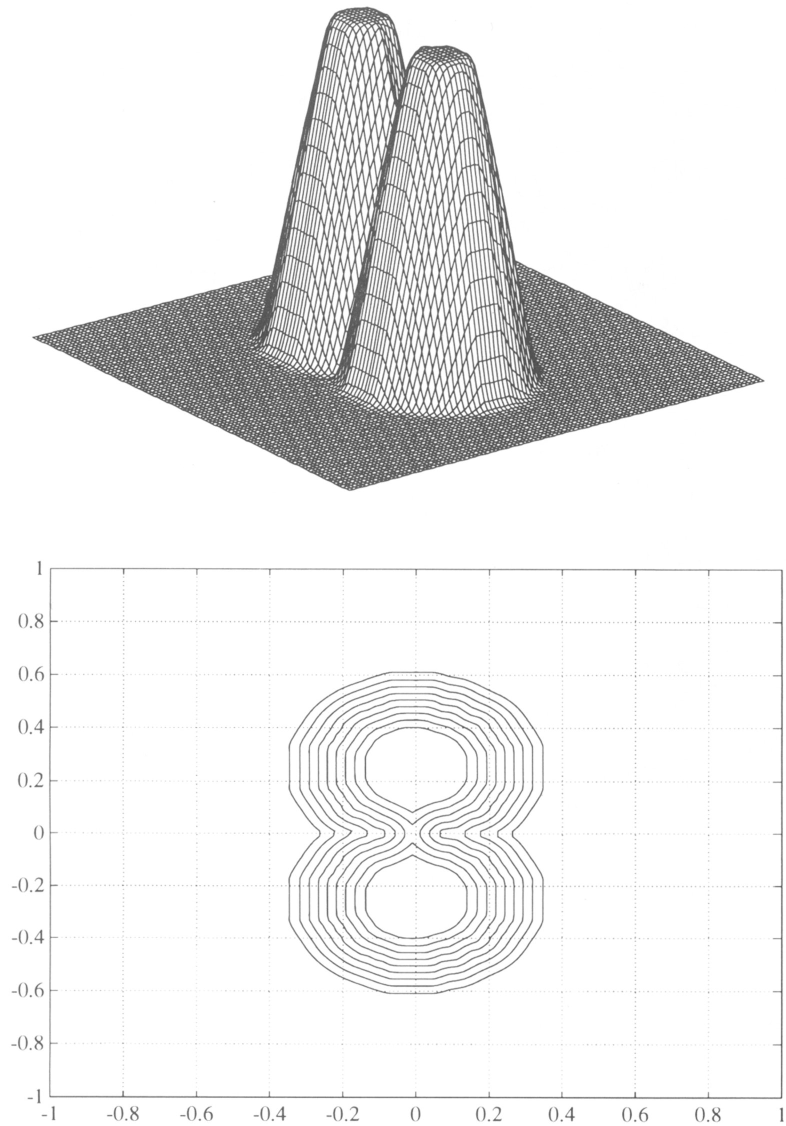

FIGURE 3

Figures 2, 3 show that curves are in general poorly resolved on a uniform regular grid. Of course, better resolution can be obtained by the use of adaptive grids.

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